1. LOCAL ENERGY ESTIMATE

Theorem 1. *Suppose that $u : \mathbb{R}^n \times [0, +\infty)$ is a solution to the wave equation $u_{tt} = c^2 \Delta u$. Given $\rho > 0$, we define the energy*

$$E(t) = \int_{B(x_0, \rho - ct)} |u_t|^2 + c|\nabla u|^2 dx. \tag{1}$$

Then, $E'(t) \leqslant 0$ holds.

See Lemma 5.11 in the textbook.

2. GLOBAL CAUCHY PROBLEM

Let $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ be a solution to the following Cauchy problem

$$u_{tt} = \Delta u, \qquad \text{in} \quad \mathbb{R}^n \times [0 + \infty), \tag{2}$$

$$u = g, \ u_t = h \qquad \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \tag{3}$$

Then, we define the spherical means by

$$U(x; r, t) = ⨍_{\partial B(x,r)} u(y, t) dy. \tag{4}$$

In addition,

$$G(x; r) = ⨍_{\partial B(x,r)} g(y) dy, \qquad H(x; r) = ⨍_{\partial B(x,r)} h(y) dy. \tag{5}$$

Lemma 2 (Euler-Poisson-Darboux equation)**.** *Given a fixed $x \in \mathbb{R}^n$, $U(x; r, t)$ is smooth on $(r, t) \in [0, +\infty) \times [0, +\infty)$ and the following hold*

$$U_{tt} = U_{rr} + \frac{n-1}{r} U_r, \qquad \textit{in} \quad (0, +\infty) \times [0 + \infty), \tag{6}$$

$$U = G, \ U_t = H \qquad \textit{on} \quad (0, +\infty) \times \{t = 0\}. \tag{7}$$

Therefore, by using the d'Alembert formula, we can obtain the Kirchhoff's formula.

Theorem 3. *If $n = 3$, then the solution is given by*

$$u(x, t) = ⨍_{\partial B(x,t)} th(y) + g(y) + \langle \nabla g(y), y - x \rangle dy. \tag{8}$$

Then, we can obtain the Poisson's formula.

Theorem 4. *If $n = 2$, the solution is given by*

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} (1 - |y-x|^2 t^{-2})^{-\frac{1}{2}} [th(y) + g(y) + \langle \nabla g(y), y - x \rangle] dy. \tag{9}$$

For details, see chapter 5.8 in the textbook.

3. Energy method

We begin by observing the following lemma.

Lemma 5. *Let $u : [0,1] \to \mathbb{R}$ be a smooth function with $u(0) = 0$. Then,*

$$|u(x)|^2 \leqslant \int_0^1 |u'(y)|^2 dy, \tag{10}$$

holds for all $x \in [0,1]$.

Proof. Since we have

$$u(x) = u(x) - u(0) = \int_0^x u'(y) dy, \tag{11}$$

the Holder inequality shows

$$|u(x)|^2 \leqslant \left(\int_0^x u'(y) dy \right)^2 \leqslant \left(\int_0^x |u'(y)|^2 dy \right) \left(\int_0^x dy \right) \leqslant \int_0^1 |u'(y)|^2 dy. \tag{12}$$

□

Therefore, we establish the following estimates.

Proposition 6. *Suppose that $u(x,t)$ satisfies $u_{tt} = u_{xx}$ for $(x,t) \in [0,1] \times [0,+\infty)$, and $u(0,t) = u(1,t) = 0$ for $t \geqslant 0$. Then, the following holds*

$$|u(x,t)|^2 \leqslant \int_0^1 |u_t(y,0)|^2 + |u_x(y,0)|^2 dy. \tag{13}$$

Proof. Let us recall the energy

$$E(t) = \int_0^1 |u_t(y,t)|^2 + |u_x(y,t)|^2 dy. \tag{14}$$

Then, as the previous lecture notes, we have $E(t) = E(0)$. Therefore, the lemma above yields the desired result. □

For the higher dimensions, we recall the L^p norm over a domain $\Omega \subset \mathbb{R}^n$

$$\|u\|_{L^p} = \int_\Omega |u(x)|^p dx. \tag{15}$$

Let us state the Sobolev inequalities. We will study them in near future.

Theorem 7. *Suppose that $u : \Omega \to \mathbb{R}$ is a smooth function with the zero Dirichlet data, namely $u(x) = 0$ on $\partial\Omega$. Then, there exist some constant C depending on n, p such that*

$$\|u\|_{\frac{np}{n-p}} \leqslant C\|\nabla u\|_p \qquad \text{for} \quad p < n,$$

$$\sup_\Omega |u| \leqslant C|\Omega|^{\frac{p-n}{np}} \|\nabla u\|_p \qquad \text{for} \quad p > n,$$

where $|\Omega|$ denote the volume of Ω.